Slim Bunny

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On the Regularity and Fractal Dimension of Iterated Metric Deformation Operators in Log-Spherical Manifolds

By SUM BUNNY

Abstract We define a class of coordinate-warped manifolds constructed via the recursive perturbation of a log-spherical metric. Let A be a compact annular domain in 3D space. We analyze the limit properties of an iterative operator T acting on the coordinate chart through a decaying sinusoidal vector field. We prove that while the sequence of diffeomorphisms converges uniformly, the regularity of the limit map Φ is governed by a spectral parameter. We derive conditions for the breakdown of differentiability, linking the Hausdorff dimension of the resulting codimension-one fractal hypersurface to the divergence of the infinite Jacobian product.

1. Introduction and Definitions

Classical geometric set construction often relies on affine Iterated Function Systems (IFS). Here, we propose a construction based on **Iterated Metric Deformation (IMD)**, modeled as a non-linear dynamical system on a coordinate manifold.

1.1 The Domain and Function Spaces To avoid singularities at the origin inherent to log-polar coordinates, we restrict our analysis to a compact annulus A (a bounded region in 3D space avoiding the origin). On this domain, the log-spherical chart is a smooth mapping.

We consider the space of bounded, Hölder continuous vector fields. We denote by $\| \cdot \|_{op}$ the operator norm on linear maps.

We define a **Warp Operator** W as a periodic vector field. For the canonical construction, we utilize integer-coupled sinusoids to preserve the discrete symmetry group:

$$W(x) = A \cdot \sin(\langle k, x \rangle + \delta)$$

1.2 The Iteration Scheme The sequence of maps is defined via the recurrence relation:

$$\Phi_{n+1}(x) = \Phi_n(x) + \lambda \cdot s^{-n} \cdot W(\gamma^n \cdot \Phi_n(x))$$

where:

- $s > 1$ is the decay factor (amplitude scaling),
- $\lambda > 0$ is the coupling constant,

- $\gamma \geq 1$ is the spectral scaling parameter (frequency scaling).

The **Limit Map** Φ_∞ can be written explicitly as a lacunary series in each coordinate:

$\Phi_\infty(x) = x + \lambda \sum [s^{-n} \cdot W(\gamma^n \cdot \Phi_n(x))]$ (sum from n=0 to ∞)

This structure reveals the limit map as a generalized Weierstrass-type function. The object of study is the embedded level set Σ, where F is a smooth scalar field:

$\Sigma = \{ p \in A \mid F(\Phi_\infty(\chi(p))) = c \}$

2. Convergence and Regularity Results

Theorem 2.1 (Uniform Convergence). Let the maximum norm of W be bounded by M. For any $s > 1$ and $\gamma \geq 1$, the sequence $\{\Phi_n\}$ converges uniformly to a limit Φ_∞ in the space of continuous functions C^0.

Proof. Consider the displacement series. The size of the step at iteration n is:

$\| \Phi_{n+1} - \Phi_n \| = \lambda \cdot s^{-n} \cdot \| W \| \leq \lambda \cdot M \cdot s^{-n}$

Since $s > 1$, this series is bounded by a convergent geometric series. By the Weierstrass M-test, the sequence converges uniformly to a continuous function.

Theorem 2.2 (Diffeomorphic Stability). There exists a small value $\varepsilon_0 > 0$ such that if the operator norm of the perturbation satisfies:

$\sup \| \nabla W(x) \|_{op} \leq \varepsilon_0$

and the sum of perturbations is strictly less than 1, then each step is a global diffeomorphism (a smooth, invertible map).

Proof. By the Global Inverse Function Theorem, a map $f(x) = x + g(x)$ is invertible if the gradient of g is everywhere less than 1. Sufficiently small coupling ensures this condition holds at every step.

3. Hausdorff Dimension and Fractal Analysis

The smoothness of the limit map is governed by the asymptotic behavior of the cumulative Jacobian product. We define:

$J_n(x) = \prod \| D\Phi_k(x) \|_{op}$ *(product from k=0 to n)* $J_\infty(x) = \lim J_n(x)$ (as $n \to \infty$)

Theorem 3.1 (Regularity Threshold).

- If $\gamma < s$, the limit $J_\infty(x)$ is finite. The limit map is smooth (C^1), and the surface Σ is a standard manifold.
- If $\gamma \geq s$, $J_\infty(x)$ diverges. The limit map is nowhere differentiable, and Σ becomes a fractal hypersurface.

Lemma 3.2 (Non-Degenerate Frequency Growth). If the frequency vectors are linearly independent, then no subsequence of partial sums converges to a smooth function, ensuring the fractal structure is not "smoothed out" by accidental cancellations.

Theorem 3.3 (Hausdorff Dimension Upper Bound). In the fractal regime ($\gamma > s$), let c be a regular value of the field F (guaranteed for almost every c by Sard's Theorem). The Hausdorff dimension of the hypersurface Σ satisfies:

$$D_H(\Sigma) \leq 3 - \alpha$$

where $\alpha = \log(s) / \log(\gamma)$ is the Hölder exponent of Φ_∞.

Proposition 3.4 (Generic Attainment of Lower Bound). For almost all choices of phase parameters (with respect to the product Lebesgue measure on $[0, 2\pi]^N$), the Hausdorff dimension equals the upper bound:

$$D_H(\Sigma) = 3 - \alpha$$

Justification: This follows by reducing the problem to the known graph dimension results of Hunt (1998) and Barański et al. (2014) for Weierstrass functions. For generic phases, the dimension is maximal.

4. Symmetry and Motivation

Proposition 4.1 (Equivariance under Discrete Actions). Let G be a discrete subgroup of the rotation group SO(3). If the warp field W uses integer couplings compatible with G, then the operator commutes with the group action. This means the fractal limit set Σ perfectly preserves the symmetry of the base lattice.

Proposition 4.2 (Log-Radial Scale Equivariance). Scaling the radius ($r \to \alpha r$) in Euclidean space corresponds to a translation in the log-spherical chart. Since W is periodic, the operator preserves local geometry under discrete zooms.

4.3 Epistemic Geometry & Systems Theory The choice of the domain is motivated by the Schwartz-Wandell model of retinotopic cortical mapping (Schwartz, 1977). By constructing fractals in a log-polar basis, we define a geometry that prioritizes rotational and scale symmetry over translational symmetry. In systems theory, this family provides a rigorous example of a **Bounded Recursive Manifold**, having finite total deformation energy while supporting infinite topological complexity. Numerical visualization and implementation are intentionally outside the scope of this paper.

5. References

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